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# Competing Theories of Financial Anomalies

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We compare two competing theories of financial anomalies: “behavioral” theories built on investor irrationality, and “rational structural uncertainty” theories built on incomplete information about the structure of the economic environment. We find that although the theories relax opposite assumptions of the rational expectations ideal, their mathematical and predictive similarities make them difficult to distinguish. Even if irrationality generates financial anomalies, their disappearance still may hinge on *rational* learning—that is, on the ability of rational arbitrageurs and their investors to reject competing rational explanations for observed price patterns.

In this article we explore competing theories of financial anomalies. A financial anomaly is a documented pattern of price behavior that is inconsistent with the predictions of traditional efficient markets, rational expectations asset pricing theory. That theory has two characteristic features. First, investors are assumed to have essentially complete knowledge of the fundamental structure of their economy.<sup>1</sup> Second, investors are assumed to be completely rational information processors who make optimal statistical decisions. Put another way, investors in the benchmark theory have “access both to the correct specification of the ‘true’ economic model and to unbiased estimators of its coefficients” [Friedman (1979, p. 38)]. As evidence has mounted against

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<sup>1</sup> In models with a representative agent, this means that the representative investor knows the true model underlying the economy. In models with heterogeneous agents, this means that there is “consistency between individuals’ choices and what their perceptions are of aggregate choices” [Sargent (1993, p. 7)].

traditional models, researchers have created competing theories of financial anomalies by relaxing those two assumptions.<sup>2</sup>

First, and probably best known, are “behavioral” explanations relaxing the second assumption (completely rational information processing). In behavioral theories, investors suffer from cognitive biases and cannot process available information rationally [Thaler (1993)]. Consistent with the experimental results that motivate behavioral finance, the background assumption in most behavioral theories is that investors act irrationally despite having considerable knowledge about the fundamental structure of the economy.<sup>3</sup>

Shiller (1981) is an early example attributing financial anomalies to irrationality, finding evidence that stock prices move too much relative to news about future dividends. DeBondt and Thaler (1985) invoke psychological evidence to motivate a price overreaction hypothesis, and find that stocks with past extreme bad returns outperform stocks with past extreme good returns. Lakonishok, Shleifer, and Vishny (1994) present evidence that superior returns earned by portfolios based on publicly available accounting and price data are consistent with excessive extrapolation of past performance into the future. Daniel, Hirshleifer, and Subrahmanyam (1998) model overreaction and underreaction from investors’ overconfidence in their private signals and biased updating in light of public information. Barberis, Shleifer, and Vishny (1998) model the same anomalies with a representative investor subject to two cognitive biases. Hong and Stein (1999) also study overreaction and underreaction, modeling the interaction of traders who naively follow price trends and traders who naively study fundamental news.

A second set of theories maintains the complete rationality assumption, but relaxes the assumption that investors have complete knowledge of the fundamental structure of the economy. This approach exploits the distinction between “rationality” and “rational expectations.” As Friedman (1979) explains, the distinction between rationality and rational expectations is the distinction between *information exploitation* and *information availability*. Inside a rational expectations world, rational investors make optimal statistical decisions in a world about which they have all relevant structural knowledge [Kurz (1994)].<sup>4</sup> Outside a rational expectations world, rational investors

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<sup>2</sup> Researchers also continue to adjust traditional rational expectations models to better fit the data, usually by modifying standard preference structures. See, for example, Constantinides (1990), Barberis, Huang, and Santos (1999), and Campbell and Cochrane (1999). Since these models retain both assumptions described above and thus, arguably at least, are still rational expectations models, we do not deal with them here. Researchers who believe that some preferences are inherently “irrational” (e.g., habit formation) may find this distinction objectionable. Less controversial are modifications that add assumptions regarding transaction costs or information asymmetry.

<sup>3</sup> Subjects exhibit cognitive biases in psychological experiments despite their ability to observe relevant data-generating processes. See, for example, Grether (1980) who finds evidence of cognitive biases in an incentive-compatible environment with observable bingo-cage data-generating mechanisms.

<sup>4</sup> As Kurz (1994, pp. 877–878) states: “[T]he theory of rational expectations in economics and game theory is based on the premise that agents know a great deal about the basic structure of their environment. In economics, agents are assumed to have knowledge about demand and supply functions, of how to extract

still make optimal statistical decisions, but they lack critical structural knowledge. “Rational structural uncertainty” models, as we refer to them, generate financial anomalies from mistakes or risk premiums that result from this incomplete information.

Merton (1987), for example, presents a model of capital market equilibrium where a given investor has information about only a subset of all securities, showing why, for example, the small-firm effect might arise. Lewis (1989) demonstrates how dollar forecast errors during the 1980s could have resulted from investors’ prior beliefs that the change in U.S. money demand would not persist, and subsequent learning about the true process generating fundamentals. Barsky and DeLong (1993) and Timmerman (1993) study rational investors who must estimate an unknown dividend growth rate, and show how learning can generate stock market volatility. Kurz (1994) presents an intricate theory of expectations formation under the assumption that agents do not know the structural relations of the economy. Morris (1996), following Miller (1977), presents a model where different Bayesian prior beliefs about an asset’s expected cash flows lead to the patterns of underperformance associated with initial public offerings (IPOs). Zeira (1999) models an economy in which changes in market fundamentals last for an unknown period of time, showing how market booms and subsequent crashes could result from rational investors’ attempt to learn about these structural changes. Lewellen and Shanken (2002) study Bayesian investors who must estimate valuation-relevant parameters, showing how estimation causes asset prices to exhibit predictability, excess volatility, and deviations from the capital asset pricing model (CAPM). Anderson, Hansen, and Sargent (1999), Hansen, Sargent, and Tallarini (2000), and Hansen, Sargent, and Wang (2000) present models where agents do not know the true data-generating process and attempt to apply robust decision rules.

In exploring the nature of these competing theories, we stress their deviations from the rational expectations ideal. We first analyze their explanatory power using simple models where representative investors must estimate an unknown valuation-relevant parameter. We use the cognitive biases of “conservatism” and the “representativeness heuristic” to motivate two behavioral models. We use Bayesian change-point analysis to motivate a rational structural uncertainty model.<sup>5</sup> We apply these models to the evidence on two important financial anomalies: overreaction and underreaction. We demonstrate how these anomalies arise in each theory, and why the behavioral and rational theories are hard to distinguish. Distinguishing the theories is hard because of underemphasized features of the empirical evidence, and because

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present and future general equilibrium prices, and about the stochastic law of motion of the economy over time. . . . [T]hese agents possess ‘structural knowledge’” (emphasis in original).

<sup>5</sup> Experimental results suggest that each of these models may have substantial explanatory power. See El-Gamal and Grether (1995). For recent experimental results related to the detection of structural change, see Massey and Wu (2000).

of mathematical similarities between the theories. Empirically overreaction and underreaction arise in different kinds of environments—overreaction after periods of longer-run recent performance and underreaction after very recent extreme performance or unusual firm events. These environments fit well with the reasons for overreaction and underreaction in both theories. Mathematically we find that the rational structural uncertainty model shares some essential features of behavioral models despite its completely Bayesian foundations—heavy weighting of old data and prior opinion in some cases, and heavy weighting of recent data and excessive certainty in others.<sup>6</sup>

We next turn to the implications of each theory for the long-term disappearance of financial anomalies. An inquiry into the disappearance of financial anomalies is essentially an inquiry into the roles that learning and arbitrage play in each theory. If rational structural uncertainty causes financial anomalies, then their disappearance hinges on the ability of rational investors to become better calibrated to the structural features of the data. This is a non-trivial task in the short run even if the economy's structural features remain stable. If those features are themselves changing in unpredictable ways, learning of this type may be impossible. Our examination of rational learning can be viewed as a special case of a large body of research on convergence to rational expectations equilibrium. This literature has made it clear that rational expectations equilibrium will not necessarily just “happen,” even if agents have the chance to learn their way to that equilibrium. [See, e.g., Blume and Easley (1982), Bray and Savin (1986), and Bray and Kreps (1987)].

If irrationality causes financial anomalies, their disappearance still may hinge on *rational* learning—that is, on the ability of rational arbitrageurs and their investors to reject competing rational explanations for observed price patterns.<sup>7</sup> Irrationality-induced anomalies cannot survive the presence of rational arbitrageurs unless there are “limits of arbitrage” that prevent the effectiveness of rational bets against mispricing. The most compelling limits of arbitrage arguments hinge on the short horizons of arbitrageurs. The limits of arbitrage literature suggests that rational arbitrageurs may be unable to credibly convey their strategies to rational investors, and therefore may be unable to keep funds committed to arbitrage [see Shleifer and Vishny (1997)]. In some cases, arbitrageurs may even be unable to convince *themselves* that exploitable mispricing exists. Either way, the limits of arbitrage hinge on the difficulty that arbitrageurs and/or their investors have in reject-

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<sup>6</sup> The third behavioral effect—excessive certainty—is known in the behavioral literature as “overconfidence” [see Odean (1998), Daniel, Hirshleifer, and Subrahmanyam (1998)]. We show that the rational structural uncertainty approach can deliver this effect, as well as those we study more thoroughly below.

<sup>7</sup> This assumes, of course, that arbitrageurs have identified the anomaly in the first place. The discussion of large-sample evidence of financial anomalies usually assumes that arbitrageurs could have detected the anomaly long before it was identified in the academic literature using advanced statistical techniques, powerful computers, and very large datasets [see Lakonishok, Shleifer, and Vishny (1994)]. Future research shedding greater light on what was actually knowable about financial anomalies through the sample periods would be quite interesting.

ing alternative competing rational explanations for price behavior in favor of behavioral explanations that would justify strong commitments to arbitrage. When rational explanations are easy to distinguish—both by rational arbitrageurs and their investors—the limits to effective arbitrage are likely to be quite small, and irrationality-induced anomalies are unlikely to survive.

The article continues as follows: Section 1 presents our illustrative behavioral and rational structural uncertainty models. Section 2 shows how both theories explain the appearance of overreaction and underreaction, and explores the problem of distinguishing them given the empirical data. Section 3 explores the implications of the competing theories for the disappearance of financial anomalies, highlighting the roles of learning and arbitrage in each. Section 4 concludes.

## 1. Models

### 1.1 The assets and the representative investors

We use simple models to illustrate the competing theories. At the beginning of each period  $t$  a single, one-period risky asset comes into existence, denoted  $A_t$ . The asset pays  $x_t$  at the end of period  $t$  and then goes out of existence. We assume that  $x_t$  is normally distributed with mean  $\mu_t$  and variance  $\sigma^2$ . The representative investor (who may be either irrational or rational, as we discuss further below) is risk neutral and values each period's asset at its expected payoff,  $\mu_t$ , called the *valuation-relevant parameter*. The representative investor does not know the value of  $\mu_t$ .

The key structural feature of the economy (about which we assume the behavioral investor is informed, but the structural uncertainty investor is not) is the *stability* of  $\mu_t$ . Call  $\mu_t$  “stable” if it is time invariant, that is, if  $\mu_t = \mu, \forall t$ . Call  $\mu_t$  “unstable” if it varies through time. For simplicity and tractability, we assume that at any time  $t = n$ ,  $\mu$  has changed at most one time in the last  $n$  time periods (though perhaps not at all). Complete structural knowledge entails knowledge as to whether  $\mu_t$  is stable or unstable; and if  $\mu_t$  is unstable, the location of the change point  $r \in \{1, \dots, n\}$ .

Our asset structure is a rather special one, abstracting from the multi-period payoffs and risk preferences that enter more traditional asset pricing models.<sup>8</sup> This structure is useful, however, in focusing attention on the consequences for estimators of cash-flow relevant parameters of cognitive biases and structural uncertainty. The most vigorous debate between adherents of rational and behavioral finance concerns the extent to which investor beliefs about valuation-relevant parameters and payoff structures should be characterized as biased or not. Far less debate concerns whether or not investors

<sup>8</sup> Risk neutrality, for example, has obvious benefits for model tractability. But combined with the simple asset framework, it also allows for sharper focus on the consequences for expectations formation of cognitive biases and rational concern with structural uncertainty. As we argue here, however, most of the behavioral-rational debate hinges on these expectations formation effects rather than differing models of risk preferences.

are properly solving the intertemporal optimization problems that characterize our most elegant asset pricing models. For example, behavioral models such as those presented by Barberis, Shleifer, and Vishny (1998) and Daniel, Hirshleifer, and Subrahmanyam (1998) study risk-neutral irrational traders facing quite simple asset structures. The seminal behavioral finance article—DeBontd and Thaler (1985)—presents no formal theory, but focuses on the observed tendency to overweight recent data in estimating the expected return on winners and losers. Our approach makes it far easier to compare ways in which both rational and behavioral models use prior beliefs, older data, and newer data to generate estimates of valuation-relevant parameters that can lead to anomalous asset price behavior [see also Heaton (1999)].<sup>9</sup> At the same time, we recognize the limitations of our simple framework. Our goal, however, is not to develop satisfactory new behavioral and rational asset pricing models, but to illustrate the difficulties presented by a rational-behavioral debate that centers on prior and data usage.

## 1.2 Irrational investors subject to the representativeness heuristic

Many experiments show that subjects expect key population parameters to be “represented” in any recent sequence of generated data, a phenomenon now known as the “representativeness heuristic” [Kahneman and Tversky (1972)]. Formulations of the representativeness heuristic in behavioral finance have fixed on the tendency of experimental subjects to overweight recent evidence, ignoring base rates and older evidence that would otherwise moderate beliefs.<sup>10</sup> We model this effect by assuming that the investor subject to the representativeness heuristic ignores prior beliefs completely and uses only recent payoffs to make estimates of  $\mu_t$ . Assume  $\mu$  is stable. Then at the beginning of period  $t = n + 1$  the representative investor employing the representativeness heuristic<sup>11</sup> does not know the value of  $\mu$ . However, she does know the realized payoffs of all prior assets,  $A_1, \dots, A_n$ . The optimal way to learn about  $\mu$  given its stability would be to use all the payoffs, applying Bayes’ rule as shown below. We assume instead (and quite arbitrarily) that the representativeness heuristic leads the investor to consider only the most recent half of the available payoffs, ignoring prior beliefs and older

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<sup>9</sup> Still we see no reason why the results presented here would not generalize to more complicated asset structures, precisely because they focus on a necessary component of any asset pricing problem: estimating valuation relevant parameters or state payoffs. Every asset pricing model we know of—regardless of its assumptions about intertemporal trade-offs or multiperiod payoffs—requires some investor knowledge about payoffs and parameters. To the extent that prior beliefs and historical data play a role in investor estimates of these, our results should be relevant.

<sup>10</sup> DeBontd and Thaler (1985) were the first to use this approach in academic finance, and more recent work by Lakonishok, Shleifer, and Vishny (1994) and Barberis, Shleifer, and Vishny (1998) appeals to the same psychological phenomenon.

<sup>11</sup> Using “representative” in reference to our investor and “representativeness” for our cognitive bias is unfortunate, but we stick to the standard terminology used in finance and psychology.

payoffs (which together can be thought of as base rates). Formally, the irrational investor using the representativeness heuristic employs the following estimator:

$$\hat{\mu}_{Beh, RH} = \bar{x}_{n/2}, \tag{1}$$

where  $\bar{x}_{n/2}$  is the mean of the most recent  $n/2$  payoffs, *Beh* denotes “behavioral,” and *RH* denotes the “representativeness heuristic.” Thus the irrational investor estimates the current value of  $\mu$  by averaging the last  $n/2$  payoffs from assets  $A_{n-n/2+1}, \dots, A_n$ , believing that the most recent payoffs are *sufficient* to learn about  $\mu$ .<sup>12</sup>

### 1.3 Irrational investors subject to conservatism

Conservatism is a documented deviation from Bayesian judgment where base rates (prior beliefs and/or older data) receive excessive weight and new data are underweighted [see Edwards (1968)]. Because conservatism is in some sense the opposite of the representativeness heuristic, behavioral models invoke its operation as occurring at different times in response to different kinds of data [see Barberis, Shleifer, and Vishny (1998)].

It is easiest to develop a model of conservatism by first considering the optimal Bayesian solution to the problem of estimating  $\mu$  when it is known to be stable. Assume that the Bayesian investor at the beginning of period  $t = n + 1$  does not know the value of  $\mu$ . He does know the realized payoffs of all prior assets,  $A_1, \dots, A_n$  and can use these to estimate  $\mu$  using Bayes’ theorem. Given the payoff structure of the assets, the likelihood for the realized past payoffs (assuming further that the asset payoffs are independent), given  $\mu$  and  $\sigma$ , is normal:

$$l(x_1, \dots, x_n | \mu, \sigma) \propto (\sigma)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot \sum_{i=1}^n (x_i - \mu)^2\right).$$

Let  $p(\mu, \sigma)$  denote the investor’s prior beliefs. Assuming a simple conjugate setup [see DeGroot (1970), Gelman et al. (1995)], these beliefs have the product form  $p(\mu, \sigma) = p(\mu | \sigma^2) \cdot p(\sigma^2)$ , where  $p(\mu | \sigma^2)$  is conditionally normal and  $p(\sigma^2)$  is scaled inverse  $\chi^2$ :

$$\begin{aligned} \mu | \sigma^2 &\sim N(\mu_0, \sigma^2 / \kappa_0) \\ \sigma^2 &\sim Inv - \chi^2(\nu_0, \sigma_0^2). \end{aligned}$$

The marginal distribution for  $\mu$  is obtained by integrating the joint posterior with respect to  $\sigma^2$ . The resulting marginal is in the form of a Student’s

<sup>12</sup> Nothing important changes if  $\mu$  is unstable. In that case, the investor discards payoffs from before the change completely (because he knows the location of the change point), but otherwise estimates  $\mu$  by way of the estimator in Equation (1).



$t$ -distribution. The risk-neutral Bayesian investor will be interested in the mean of this marginal distribution, given by

$$\hat{\mu} = \frac{\kappa_0 \mu_0 + n \bar{x}_n}{\kappa_0 + n}. \tag{2}$$

In exploring the respective weighting of data and prior beliefs, it is helpful to rewrite Equation (2) as the weighted average of the prior mean and the sample mean:

$$\hat{\mu} = \left( \frac{\kappa_0}{\kappa_0 + n} \right) \mu_0 + \left( \frac{n}{\kappa_0 + n} \right) \bar{x}_n. \tag{3}$$

The weights are functions of the number of observations,  $n$ . Using this estimator for  $\mu$ , the price of the asset follows.

One embodiment of the conservatism bias is overweighting of prior beliefs and underweighting of the available data. We thus model the conservative investor as estimating  $\mu$  using the following “conservative” version of Equation (3):<sup>13</sup>

$$\hat{\mu}_{Beh, C} = \left( \frac{c}{c + n} \right) \mu_0 + \left( \frac{n}{c + n} \right) \bar{x}_n, \tag{4}$$

where  $c > \kappa_0$  and subscript  $C$  denotes “conservatism.” The estimator in Equation (4) with  $c > \kappa_0$  always puts higher than optimal weight on the prior belief given the above assumptions.<sup>14</sup>

### 1.4 Rational investors with structural uncertainty

There are two crucial differences between the irrational investors and the rational investor. First, unlike irrational investors, the rational investor employs fully Bayesian methods. Second, unlike the irrational investors, the rational investor does not know whether or not  $\mu_t$  is stable so his (Bayesian) estimator for  $\mu$  must incorporate this ignorance.

Recall that we consider  $\mu_t$  “unstable” if it might vary through time, and that at any time  $t = n$ , the investor considers that  $\mu$  changed at most one time in the last  $n$  time periods (though perhaps not at all) at an unknown (to the rational investor) “change point,”  $r \in \{1, \dots, n\}$ . That is, the investor assumes that the payoffs,  $x_1, \dots, x_n$ , were generated by mean  $\mu_A$  for

<sup>13</sup> It is important to note that while Equation (4) captures the essential feature of the conservatism bias (heavy weighting of prior opinion), it is also consistent (in a formal sense) with a certain parameterization of rational Bayesian beliefs. In a laboratory setting with induced priors, the experimenter is able to rule out this parameterization so that heavy weighting of the prior cannot be “rationalized.” Still, using a formally Bayesian structure to model irrationality outside the laboratory presents some difficult philosophical issues [see Winkler and Murphy (1973)]. We address some of these issues in our concluding remarks.

<sup>14</sup> Again, nothing important changes if  $\mu$  is unstable. In that case, the irrational investor does discard payoffs from before the change completely, since we assume that the irrational investor uses his structural knowledge. Given the payoffs he uses, however, the irrational investor applies Equation (4) and thus exhibits conservatism.

$t \in \{1, \dots, r\}$  and  $\mu_B$  for  $t \in \{r+1, \dots, n\}$ . Thus  $r$  denotes the payoff after which payoffs are generated by the new mean,  $\mu_B$ . The state of “no change” is  $r = n$ . In that case the investor believes that  $\mu_A$  generated all payoffs up to time  $t = n$ .

At the beginning of period  $t = n + 1$  the rational investor does not know the value of  $\mu_{n+1}$ , but he does know the realized payoffs of all prior assets,  $A_1, \dots, A_n$ . He can use the payoffs of these prior assets to estimate  $\mu_{n+1}$ . Because his estimator must account for the possibility of a change from  $\mu_A$  to  $\mu_B$ , he requires a posterior distribution over the possible change points (the point of the change, if any, from  $\mu_A$  to  $\mu_B$ ). Smith (1975) shows how to generate this posterior probability distribution in the single change-point case. The rational investor first specifies a prior distribution over the possible change points. Including the possibility of no change,  $r = n$ , there are  $n$  possible change points. That is, the change either occurred at one time  $t \in \{1, \dots, n-1\}$  or it did not occur at all. Creating a prior probability distribution over the possible change points requires the assignment of prior probability to each possible change point such that  $p_0(1) + p_0(2) + \dots + p_0(n) = 1$ . Subscript 0 denotes a prior probability specified before any payoffs are observed. Subscript  $n$  denotes a posterior probability where  $n$  payoffs have been observed. We assign a uniform prior over the possible change points,  $r \in \{1, \dots, n\}$ . This uniform prior is in fact an “informative prior” that models a fairly strong belief in the potential instability of  $\mu$ .<sup>15</sup> We assign informative prior beliefs to  $\mu_A$  and  $\mu_B$ , and a (degenerate) prior belief that they are independent conditional on  $\sigma^2$ . The posterior distribution for the change points is then

$$p_n(r) = \frac{p(x_1, \dots, x_n|r)p_0(r)}{\sum_r p(x_1, \dots, x_n|r)p_0(r)}, \tag{5}$$

where

$$p(x_1, \dots, x_n|r) = \int_{\mu_A, \mu_B, \sigma} p(x_1, \dots, x_n|r, \mu_A, \mu_B, \sigma) \times p_0(\mu_A|\sigma)p_0(\mu_B|\sigma)p_0(\sigma) d\mu_A d\mu_B d\sigma. \tag{6}$$

Appendix A sets forth the derivation of the posterior probability distribution for the change points. Smith (1975) shows how to derive marginal distributions for  $\mu_A$  and  $\mu_B$  using that distribution. These are given by

$$p_n(\mu_i) = \sum_r p_n(\mu_i|r)p_n(r) \quad (i = A, B). \tag{7}$$

<sup>15</sup> Assigning identical probability to each possible change point means that the “no change” point  $r = n$  receives prior probability  $1/n$ , while the total probability assigned to the event “some change” is  $(n-1)/n$ .

Each  $p_n(\mu_i|r)$  is a posterior distribution for  $\mu_A$  or  $\mu_B$ , conditioned on the change having occurred at a given change point,  $r$ . The final posterior distribution is the weighted average of these conditional posterior distributions. The weights are the posterior probabilities of the change points. The investor's asset pricing problem requires a marginal distribution for  $\mu_{n+1}$  at the beginning of time  $n + 1$ . We abstract from the inherent forecasting problem<sup>16</sup> and assume that the investor uses his marginal distribution for  $\mu_n$ :

$$\sum_{r=1}^{n-1} p_n(\mu_B|r)p_n(r) + p_n(\mu_A|r = n)p_n(n).$$

Note that the estimator reflects the rational investor's lack of knowledge as to which of  $\mu_A$  or  $\mu_B$  generated the payoffs at time  $t = n$ . The first term reflects the possibility that there may have been a change from  $\mu_A$  to  $\mu_B$  at or after time  $t = 1$ . In this case,  $\mu_B$  is the current parameter value at time  $t = n$ . Note, however, that in estimating the value of  $\mu_B$  (in the event it is the current parameter), the rational investor must consider each possible scenario, from the possibility that all payoffs after the first were generated by  $\mu_B$  ( $r = 1$ ), to the possibility that only the last payoff point was generated by  $\mu_B$  ( $r = n - 1$ ). The second term reflects the possibility that there may have been no change ( $r = n$ ), in which case  $\mu_A$  generated all payoffs through time  $t = n$ . In Appendix B we show that the mean of this distribution, given our assumptions, is

$$\hat{\mu}_n = \sum_{i=1}^{n-1} p_n(i) \left[ \frac{(n-i)}{(n-i) + \kappa_0} \bar{x}_{n-i} + \frac{\kappa_0}{(n-i) + \kappa_0} \mu_0 \right] + p_n(n) \left[ \frac{n}{n + \kappa_0} \bar{x}_n + \frac{\kappa_0}{n + \kappa_0} \mu_0 \right], \tag{8}$$

where  $\bar{x}_{n-i}$  denotes the mean of the  $n - i$  most recent payoff observations (that is, all payoffs after the change point  $i$  on which the mean is conditioned) and the  $p_n(\cdot)$  are as defined above. Just as in the stable case of Equation (3), the estimator in Equation (8) is written as the weighted average of sample means and the prior mean. In fact, it is easy to see that Equation (8) nests, as it must, the estimator in Equation (3). When the posterior probability of "no change"  $p_n(n)$  equals 1.0, only the last term remains, and that is just Equation (3). In the estimator of Equation (8), there are  $n - 1$  possible sample means entering the estimator of  $\mu_B$ —one for each possible estimator of  $\mu_B$  given that a change occurred at some point  $r \in \{1, \dots, n - 1\}$ —and 1 possible sample mean entering the estimator of  $\mu_A$  if there was no change, that is,  $r = n$ .

<sup>16</sup> Technically the investor requires an estimate of  $\mu_{n+1}$ , not  $\mu_n$ . Abstracting from this problem introduces a very small order error, but allows for a more tractable model.

## **2. Explaining Financial Anomalies**

We now use the three models developed above to demonstrate how the competing theories can explain two well-known anomalies—overreaction and underreaction—and why the behavioral and rational theories are difficult to distinguish.

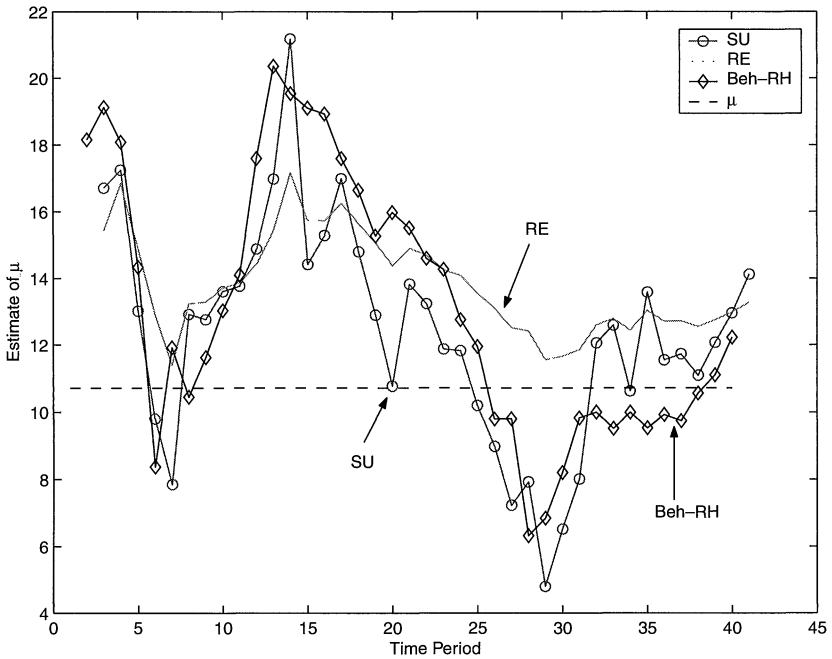
### **2.1 Overreaction and underreaction**

“Overreaction” refers to the predictability of good (bad) future returns from bad (good) past performance [see, e.g., DeBondt and Thaler (1985), Lakonishok, Shleifer, and Vishny (1994)]. Overreaction has been found using portfolio formation strategies that sort on proxies for recent performance in a given direction (e.g., recent years of good or bad earnings or returns). Consider the superiority of value stock investment strategies over growth stock investment strategies documented by Lakonishok, Shleifer, and Vishny (1994). “Value” stocks that outperform growth stocks in their study were recent (last five years) poor performers in terms of earnings, cash flow, and sales growth, while “growth” or “glamour” stocks were consistent good performers over the same horizon. In an earlier study, DeBondt and Thaler (1985) sorted firms on the basis of three- to five-year past returns, sorting firms into loser and winner portfolios. In both types of studies, later performance suggests that prices placed too much weight on this past performance, that is, that prices “overreacted” to the recent good and bad past performance.

“Underreaction” refers to the predictability of good (bad) future returns from good (bad) past performance [see, e.g., Jegadeesh and Titman (1993), Michaely, Thaler, and Womack (1995), Chan et al. (1996)]. Underreaction has been found using portfolio formation strategies that sort on proxies for extremeness of some sort (including unusual firm events). Consider the superiority of momentum strategies documented by Chan et al. (1996). They sort firms based on standardized unexpected earnings, extreme recent returns, and changes in analysts’ forecasts. On each measure, winners continue to be winners in the immediate future, while losers continue to be losers. The authors find no significant evidence of price reversals. The drift to new price levels is permanent, consistent with the existence of evidence in the extremeness of an actual change in a valuation-relevant parameter that investors recognized only slowly.

### **2.2 Overreaction and underreaction in the behavioral and rational models**

At first glance, overreaction and underreaction present a considerable challenge to any theory. Nevertheless, both of the competing theories can explain these results, as illustrated by the simple behavioral models and the rational models developed above.



**Figure 1**  
**Overreaction**

Consider overreaction. The evidence suggests that overreaction can occur when investors put too much weight on recent performance. Figure 1 illustrates this effect. In Figure 1,  $\mu$  is stable at  $\mu_A$  for the entire simulated sample period of 40 observations. The benchmark rational expectations estimator is given by RE, reflected in Equation (3). That estimator reflects both complete rationality and is calculated at each point assuming the correct state of stability. SU is the rational structural uncertainty estimator of Equation (8). That estimator reflects the uncertainty regarding possible structural change in the data. Beh-RH is the representativeness heuristic estimator from Equation (1). That estimator uses only the last half of the data for estimation.<sup>17</sup>

Figure 1 reflects a typical sample path for these three estimators, given payoff realizations. Note first that even the RE estimator exhibits a form of “overreaction,” since estimation error in any given sample path will force that estimator above the true value for recent “good” observations, and below

<sup>17</sup> The sample path presented in Figure 1, as well as the one presented later in Figure 2, were generated as follows. We first specified the following prior parameters:  $\mu_0 = 10$ ,  $\kappa_0 = 1$ ,  $\nu_0 = 40$ ,  $\sigma = 15$ . Then we drew from the investor’s prior beliefs sample realizations of  $\mu_A$ ,  $\mu_B$ , and  $\sigma_0$ . Each of these two sets was used to generate sample realizations of length 40. Figure 1 presents a sample path in which the unknown mean equals 10.7, the unknown standard deviation equals 12.5, and no change has occurred. Figure 2 presents a sample path in which the standard deviation equals 13.7 and a break occurred after period 20 moving from 11 to 0.3.

the true value for recent “bad” observations. This illustrates the observation of Timmerman (1993) and Lewellen and Shanken (2002) that even rational learning will exhibit forms of overreaction (and excess volatility) on the way to convergence.

Now consider the behavioral estimator, Beh-RH. The extreme overreaction to recent data that occurs from using this estimator is apparent by comparison to RE. Variation around the true value of  $\mu$  is caused by the effect of recent payoffs in one direction or the other. However, these effects are exacerbated in the Beh-RH estimator, compared to RE. In the RE estimator, the effect of recent data is moderated both by the effect of the prior and the older data (recall that the irrational investor employing the representativeness heuristic is ignoring both old data and any prior). Invoking the representativeness heuristic, behavioral theories can posit an irrational investor who believes that recent data is sufficient to describe the underlying data-generating process.

We next turn to the rational structural uncertainty estimator, SU. This estimator also exhibits extreme overreaction compared to RE. As we show mathematically below, more extreme variation around the true value of  $\mu_A$  is—as with the estimator Beh-RH—caused by the effect of recent payoffs in one direction or the other. In the SU estimator, heavy weight on recent data is a reaction to the concern with structural change. Whenever that change does not occur, the weight placed on recent data will be too high. This will result in a pattern of overreaction strikingly similar to that caused by the representativeness heuristic.

Figure 2 illustrates the underreaction effect. The evidence suggests that underreaction reflects extremeness of some sort, particularly a change in some underlying valuation-relevant parameter. Empirical proxies include standardized unexpected earnings, extreme recent returns, and changes in analysts’ forecasts. Underreaction appears to be associated with a failure to fully incorporate the price implications of this change in a valuation-relevant parameter. In Figure 2,  $\mu_A$  is stable until observation 20, changing then to  $\mu_B$  for the remaining 20 periods. The benchmark rational expectations estimator is again given by RE, reflected in Equation (3). That estimator reflects both complete rationality (being a Bayesian calculation) and complete structural knowledge: the estimator is calculated at each point assuming the correct state of stability of  $\mu_A$  through observation 20, and then with knowledge of the correct state of stability of  $\mu_B$  through the remaining simulated periods. SU is the rational structural uncertainty estimator of Equation (8). That estimator reflects the uncertainty regarding possible structural change in the data and a lack of knowledge that a change occurred at observation 20. Beh-C is the conservatism estimator from Equation (4). That estimator, by construction, places too little weight on data as we set  $c = \kappa_0 + 5$  and (correspondingly) overweights the prior.

Figure 2 reflects a sample path for these three estimators, given payoff realizations before and after a change. Note first how well the RE estimator

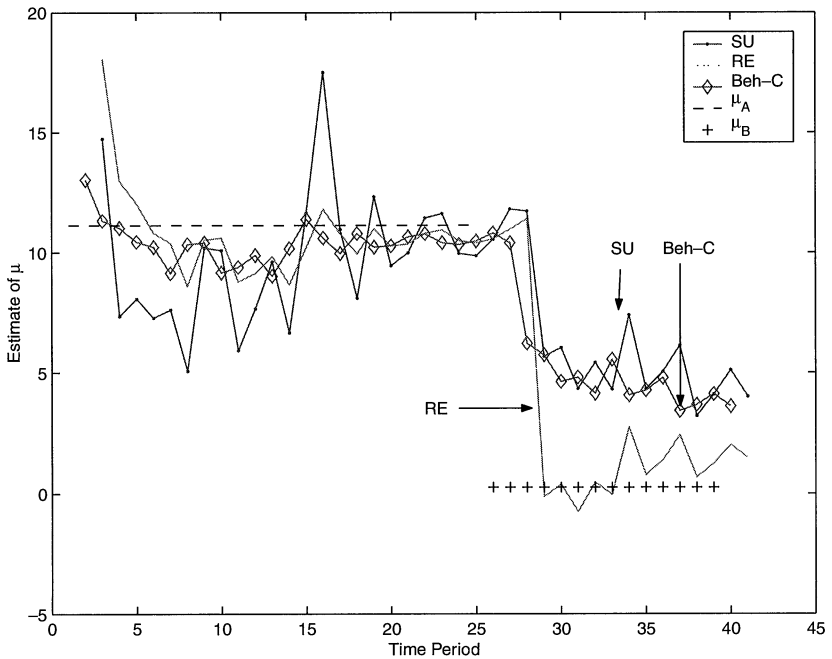


Figure 2  
Underreaction

can perform in responding to the change. There is still a form of “overreaction” caused by estimation both before and after the change from  $\mu_A$  to  $\mu_B$ . But there is no “underreaction.” The estimator moves quickly to the new level of  $\mu_B$ .<sup>18</sup>

Now consider the behavioral estimator, Beh-C, from Equation (4). The estimator appears very much like the RE estimator until the change. At the change, however, the Beh-C estimator drifts quite slowly toward the new level of  $\mu_B$  by comparison to RE. This is caused by the low weight placed on the new data, or, put another way, the excess weight placed on the prior. Invoking conservatism in response to extreme earnings or returns that exhibit subsequent drift, behavioral theories can posit an irrational investor who underweights new data by overweighting his prior beliefs.

The rational structural uncertainty estimator, SU, also exhibits drift by comparison to RE. This drift is caused by the underweighting of new data

<sup>18</sup> The examples in this section illustrate the similarity between a structural uncertainty model and a behavioral model, and the parameter values chosen (indirectly, through selection of the paths) serve this purpose. It is important to remember, however, that the RE sample path shown here is simply one path of many, and depends on the values of the unknown parameters relative to the prior mean. Not all paths would drop this fast. What matters is that the RE estimator can approach the new level much faster than SU and Beh-C, given knowledge of the break and fully Bayesian updating.

that occurs from considerable weight that remains on old data and prior beliefs. In the SU estimator, insufficient weight on new data occurs because of the incomplete information about the parameter change. When that change occurs, the weight placed on new data will be too low. This will result in a pattern of underreaction strikingly similar to that caused by conservatism.<sup>19</sup>

### **2.3 Distinguishing the theories**

The simulation results shown in Figures 1 and 2 illustrate potential behavioral and rational explanations for well-known financial anomalies. Those results also suggest that behavioral and rational explanations might be quite hard to distinguish: the patterns of overreaction and underreaction can be essentially the same. In any given model, of course, the behavioral and rational theories might be parameterized so as to be distinguishable even in the simple simulations we posit here. For example, our model of the representativeness heuristic is quite extreme, and a close look at Figure 1 suggests some ability to distinguish the Beh-RH and SU estimators on the basis of the more extreme estimates generated by Beh-RH. This is illusory, however, since an alternative model with some (albeit too little) weight on the prior could force the Beh-RH and SU estimators even closer in Figure 1. Similarly, Figure 2 suggests that the Beh-C estimator might be distinguishable from the SU estimator by its lower degree of overreaction before the change. However, a parameterization of conservatism that differentiated the weight placed on older and newer data (instead of modeling only the greater weight on the prior) would also force Beh-C and SU even closer in Figure 2.

Aside from these special cases, the general problem of distinguishing the theories at this level arises for two related reasons. First, overreaction and underreaction seems to arise in different kinds of environments—overreaction after periods of longer-run recent performance, and underreaction after very recent extreme performance or unusual firm events—and these environments fit well with the reasons for overreaction and underreaction in both theories.

Second, despite their obviously different underlying assumptions, the theories bear considerable mathematical resemblance to each other. This mathematical similarity is the driving force behind the ability of both theories to explain similar evidence, and the difficulty of distinguishing the theories with that same evidence. Distinguishing the theories empirically requires, at a minimum, that behavioral and rational structural uncertainty models make

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<sup>19</sup> These interpretations also apply easily to long-run event study evidence. Consider, for example, the evidence on the event-day and long-run returns to dividend initiations [Michaely, Thaler, and Womack (1995)]. For dividend initiations, pre-event strong operating and price performance is associated with the subsequent dividend initiation. That event is associated with a positive event-day abnormal return and subsequent positive drift. This can be interpreted as consistent with the behavioral explanation of conservatism, in particular, an underreaction to the new information contained in the initiation event. To the extent that this event reflects a transition to either a lower level of systematic risk or higher operating performance (a structural break) [see Grullon, Michaely, and Swaminathan (2001) for evidence supporting this interpretation], the rational structural uncertainty approach also can generate positive drift. Similar explanations apply to other long-run event studies.



different predictions given the available data. Ideally, given a set of information (e.g., historical returns, dividends, earnings, etc.), behavioral investors would form different expectations from rational but structurally uninformed investors and these expectations would manifest in different patterns of price behavior. These differences would provide the basis for distinguishing the theories.

Unfortunately the estimators given by Equations (1), (4), and (8) exhibit the same basic mathematical properties. Recall that the representativeness heuristic involves heavy weighting of recent data, while conservatism leads to underweighting of recent data. In the structural uncertainty model, beliefs about the stability of valuation-relevant parameters determine the respective importance in estimates of those parameters of older data, newer data, and an investor's prior beliefs. In Appendix C we show that the SU estimator involves heavy weighting of recent data (and excessive certainty) when applied in a stable environment, just as with the Beh-RH (representativeness heuristic) estimator. This is the situation (see Figure 1) when the SU estimator exhibits overreaction. It is plain from Equation (8) that the SU estimator underweights new data immediately after a change, since old (and, by definition, irrelevant) data enter the estimate, just as with the Beh-C estimator. This is the situation (see Figure 2) when the SU estimator exhibits underreaction (drift).

It is easy to see that the implications of this approach lead to a similarity with the behavioral models in the empirical environments that seem to characterize overreaction and underreaction. Explaining overreaction requires the ability to invoke heavy weighting of recent payoffs in an environment where that weighting was not justified, *ex post*. Behavioral models can invoke the representativeness heuristic. Rational models can invoke a concern with instability that will necessarily bring with it a heavy weight on recent data, as described above and proven in Appendix C. Explaining underreaction requires the ability to invoke insufficient weighting of new payoffs in an environment where those payoffs contained essential information about the new valuation-relevant parameter. Behavioral models can invoke conservatism. Rational models can invoke concern with instability. Anytime the rational investor fails to identify the change exactly, he will carry weight on old data into the postchange period. This will cause a drift that may be virtually indistinguishable from that caused by conservatism.

We can rephrase the problem more generally. In the rational structural uncertainty model, beliefs about the stability of the valuation-relevant parameter determine the *respective importance* in estimates of those parameters of older data, newer data, and the investor's prior beliefs. But these are precisely the contours of the cognitive biases—conservatism and the representativeness heuristic—that motivate the behavioral models. Thus at a basic level, the theories are hardly distinguishable, if at all, based on their use of data and prior beliefs. Investors placing low weight on new data may be acting irrationally

and displaying conservatism, but they also may be placing more weight on old data and prior beliefs in the (rational) belief that the underlying parameters might not have changed. Alternatively, investors placing heavy weight on recent data may be acting irrationally and displaying the representativeness heuristic, but they also may be placing more weight on recent data in the (rational) belief that the underlying parameters are unstable, rendering the older data less relevant to their estimates.

The mathematical similarities are not limited to heavy weighting of recent data in some cases and underweighting of new data in others. The rational structural uncertainty approach is clearly flexible enough to capture other biases as well. Consider, for example, the “excessive certainty” or “overconfidence” effect. Overconfidence is the belief that the precision of one’s information or beliefs is greater than actual. Put differently, overconfident individuals are too sure of themselves. Overconfidence has been applied in the work of Daniel, Hirshleifer, and Subramanyam (1998) and Odean (1998) by assuming that investors arrive at variance estimates that are too low.

Overconfidence arises in a structural uncertainty framework when an investor (or trader, or manager) believes that some quantity of interest may be changing through time. Consider, for example, an investor who is estimating the performance of an investment strategy by looking at the mean and variance of its returns. He receives return data through time. Now consider two types of investors. Both believe that the unknown variance does not change over time. However, the first type of investor believes that the unknown mean return of the investment strategy may have changed over the period, while the second type of investor believes that the unknown mean return to the strategy is stable through time. We show in Appendix D that the structural uncertainty of the first investor will lead him—in most cases—to have a smaller variance estimate relative to the second investor. The basic intuition of the result is that an investor who believes in stability derives his posterior beliefs regarding an unknown variance by calculating a sum of squares measure about his posterior estimate of the unknown mean, while the investor concerned with instability calculates his sum of squares about more than one sample mean as he allows for a possible change. Unless the sample size is quite small, the reduction of uncertainty due to the lower sum of squares measure leads to a lower posterior mean for the unknown variance. The “overconfidence” effect will occur simultaneously with the heavy weighting of recent data, since both arise from the belief in instability.<sup>20</sup>

In the end, the similarity of the behavioral and rational models raises the interesting speculative possibility that cognitive biases are themselves somehow related to structural uncertainty. Winkler and Murphy (1973) suggested

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<sup>20</sup> The tendency of overreaction to coincide with overconfidence in a stable environment could be applied, for example, to model market crashes and excess volatility as investors react too abruptly to a string of either good or bad asset performance.

some time ago that the conservatism effect may be an artifact of legitimate concern with nonstationarity that is hard to shake in laboratory settings.<sup>21</sup> Of course, so far as we know, not even psychologists have arrived at a unifying theory of these biases, let alone a theory to explain the origins of overconfidence [see, e.g., Brenner et al. (1996), Griffin and Varey (1996), Griffin and Buehler (1999)], so economists must remain modest in advancing their own “origin” theories. Nevertheless, the rational structural uncertainty framework does provide an economic context for many such behaviors. And it is intriguing that our prediction that overconfidence is likely to coincide with overreaction (representativeness heuristic) has been advanced in the psychology literature [Griffin and Tversky (1992)].<sup>22</sup> Testing the relation between cognitive biases and concern with structural uncertainty may prove a useful avenue for experimental economists.

### 3. Learning and Arbitrage

In this section we explore a question that seems to pervade the discussion of financial anomalies: When will financial anomalies disappear? There are two obvious means by which an anomaly might disappear. First, the behavior causing the anomaly might change as investors learn something that alters their expectations. In our simple models, anomalies arise from the incorrect use of data. In behavioral models, investors use data incorrectly because of cognitive biases. In rational structural uncertainty models, investors use data incorrectly when they are mistaken about the structure of the environment. In either case, learning might cause those anomalies to disappear. Second, arbitrage might cause the anomalies to disappear as investors who are not making mistakes place bets to exploit the beliefs of investors who are. That idea has a considerable pedigree in modern financial economics [see Friedman (1953), Fama (1965)] and remains the quintessential objection to behavioral finance.

The prospects for learning in behavioral models have not been well explored, and the “noise traders” in those models rarely learn not to employ bad investment strategies [see, e.g., Barberis, Shleifer, and Vishny (1998)]. Some researchers object to this approach. As DeLong et al. (1990b, p. 384) state:

An important objection to this approach is that [noise traders] are really dumb; they do not realize how much money they lose by chasing the trend. Why don't [noise traders] . . . learn that they are making mistakes?

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<sup>21</sup> Of course, being unable to “shake” a tendency when it is no longer appropriate is itself a form of irrationality.

<sup>22</sup> Griffin and Tversky suggest that confidence is a function of the extremeness of the evidence and its credence. Overconfidence arises when subjects focus on the extremeness of the evidence with insufficient regard to credence. Underconfidence is hypothesized to occur when the extremeness of the hypothesis is low and its credence is high. Griffin and Tversky therefore propose that overconfidence will occur when base rates are low (representativeness), while underconfidence will occur when base rates are high (conservatism), similar to the predictions made by the structural uncertainty model.

Answering the question for their own behavioral model, the authors argue that “every episode might look different to [noise] traders, and so their learning from past mistakes might be limited.” Lakonishok, Shleifer, and Vishny (1994) note the potentially confusing nature of short-term confirmation in the data for noise traders who otherwise revise their expectations. Evidence from the laboratory lends credibility to these assumptions. In general, psychologists find that learning in experiments requires immediate outcomes and clear feedback. When circumstances present neither—as with financial markets, given their delayed outcomes and noisy feedback—learning from experience may be unlikely [see Brehmer (1980)].

In the simple rational model used here, learning would occur “in the long run,” but the ease of long-run learning is partly an artifact of our modeling choices and may not be a realistic representation of potential behavior. We have posited a single change-point model, while the real world presents the possibility of multiple change points. The introduction of multiple change points (while quickly introducing intractable modeling problems) might prevent learning in most scenarios.

Of course, if the rational investor is endowed with an objectively correct prior regarding the frequency of these changes and the right likelihood functions, it is easy to show that he will converge to the rational expectations solution and that structural uncertainty-induced anomalies should disappear. Attainment of such an extreme knowledge has been studied extensively in the literature on convergence to rational expectations equilibrium. Blume and Easley (1982, p. 341; 1998, p. 99) point out that investors would need to recognize and incorporate how *their beliefs* about the unknown structural features of the economy affect the structural model of the economy. They [Blume and Easley (1984, p. 127–128)] make it amply clear, however, that the amount of knowledge required in these models is implausible. Bray and Kreps (1987, p. 622) make a similar observation:

Insisting that learning is based on correctly specified priors and conditional distributions brings us back to learning within a grand rational expectations equilibrium. It guarantees convergence of posteriors on parameter values, but merely pushes one stage back the question of how agents learn about the rational expectations equilibrium.

If investors do not recognize the effect that learning has on prices in equilibrium, Blume and Easley (1982) have shown, within a general equilibrium learning model, that convergence of beliefs is not guaranteed.<sup>23</sup> Instead,

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<sup>23</sup> The literature on learning rational expectations is extensive. Some examples are DeCanio (1979), Bray and Savin (1986), Bray and Kreps (1987), and Foster and Frierman (1990). In general, the possibility that expectations converge to the rational expectation equilibrium hinges in large part on the nature of the particular learning process and the structural features of the chosen economy.

learning may lead to arbitrary outcomes such as cycles, divergence, and even convergence to an incorrect model. Thus the learning dynamics that we study here can be viewed as a manifestation of particular paths of such beliefs. This view is consistent with Bossaerts (1995) and Marimon (1997), who emphasize that learning models can be used to explain price patterns that do not satisfy rational expectations equilibrium restrictions.

Nevertheless, it is important to ask what are the specific circumstances in which rational investors might learn that anomalies are inconsistent with their prior beliefs and information set.<sup>24</sup> While we do not address such calibration issues here [see Abarbanell and Bernard (1992)], one could ask what kind of individual firm's earnings processes could generate, in the cross section, both underreaction and overreaction. Within a structural uncertainty framework, one possible avenue would be to model valuation-relevant parameters, such as earnings growth, as in Bansal and Yaron (2000). These authors model aggregate dividend growth as an ARMA (1, 1) process, which captures both the effect of long-term stochastic trends and cyclical variations of dividend growth on asset prices. This process might fit well with the rational structural uncertainty models analyzed in this article if it captures rational investors' uncertainty regarding two components on the growth rate, but much future research is needed before reaching that conclusion.<sup>25</sup>

Arbitrage has little role to play in a purely rational structural uncertainty theory, since investors there are doing the best they can given the information they have. But arbitrage plays a key role in behavioral theories. In fact, if there are rational arbitrageurs in the economy, then irrationality-induced anomalies can survive only if something limits the effectiveness of arbitrage. The so-called noise trader or limits of arbitrage literature emerged because of this "arbitrage objection" to behavioral finance: the claim that competitive arbitrage will drive to zero any mispricing caused by behavioral traders' bad investment strategies. While this objection sounds nearly irrefutable, recent

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<sup>24</sup> Many researchers believe that the large sample statistical significance found in anomalies studies implies that investors could easily change their behavior to avoid or exploit the mispricing. For example, studies on the value-growth anomaly lead many to assume the clear superiority of value strategies. However, as Shleifer (2000, p. 103) notes, value strategies beat growth strategies only about two-thirds of the time. For post earnings announcement drift, matters are even less clear. Only about 54% of firms in the most extreme positive earnings surprise decile have positive buy-and-hold abnormal returns. Similar results hold for the extreme negative surprise returns. (We thank Arthur Kraft of the University of Rochester for providing us these data.) Calibrating in such environments may be quite difficult.

<sup>25</sup> Neither rational agents nor behavioral agents are required to learn only about fundamentals, they also can learn about returns. However, most anomalies are not present at all times in the data, implying that structural uncertainty pervades attempts to learn from returns as well. Connolly (1991), for example, uses Bayesian posterior odds analysis (which can be interpreted as the learning process of a rational investor) to show how unstable was the "weekend effect" in the 1963–1983 sample period. His results can be interpreted as reflecting rational doubt on the part of market participants about the very existence of this anomaly for significant periods of time. Of course, while rational learning within a structural uncertainty approach provides a powerful tool for the disappearance of financial anomalies, survey results demonstrate that even in academia, in which learning is presumably rational, there is still no consensus as to the sources of cross-sectional differences in average returns due to size, book-to-market, and momentum effects [see Welch (2000, Table 6)].

theoretical and empirical analyses of arbitrage have weakened its force somewhat and allowed behavioral theories to proceed with less worry [see, e.g., Pontiff (1996) and Shleifer and Vishny (1997)].<sup>26</sup>

What is not emphasized, however, is that appeals to the limits of arbitrage tend to connect behavioral finance to the structural uncertainty approach. Consider Shleifer and Vishny (1997). They point out that arbitrageurs typically speculate with other people's money and those people tend to withdraw funds after poor performance. The prevalence of performance-based arbitrage may leave the most severe episodes of mispricing unmitigated. Why investors in arbitrage funds should withdraw funds from arbitrageurs after bad performance is somewhat puzzling, however, given the obvious possibility that mispricing from which they hope to profit may simply have deepened. Shleifer and Vishny (1997), though clearly focused on the survival of irrationality-induced anomalies, justify this behavior by an appeal to *rational* structural uncertainty on the part of investors who provide capital to arbitrageurs:

Both arbitrageurs *and their investors* are fully rational. . . We assume that investors have no information about the structure of the model determining asset prices. . . Implicitly *we are assuming that the underlying structural model is sufficiently nonstationary and high dimensional that investors [who provide arbitrageurs with funds] are unable to infer the underlying structure of the model* from past returns data. . . Under these informational assumptions, individual arbitrageurs who experience relatively poor returns in a given period lose market share to those with better returns. (pp. 38, 40; emphasis added)

In other words, the key to the limits of arbitrage in Shleifer and Vishny (1997) is the existence of rational structural uncertainty on the part of their investors, not cognitive biases. This structural uncertainty causes rational arbitrageurs to have short investment horizons that prevent complete arbitrage [see also, DeLong et al. (1990a)]. But here the issue has truly come full circle. The explanation for short horizons is a form of rational structural uncertainty where arbitrageurs and/or their investors cannot be certain of the existence of arbitrageable mispricing and limit their capital commitments accordingly. This implies that if it is easy for rational arbitrageurs to reject rational explanations for particular patterns of price behavior, then it should be easy for those rational arbitrageurs to convince rational investors to provide long-term capital for arbitrage.

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<sup>26</sup> Blume and Easley (1992) examine the dynamics of wealth accumulation and ask whether natural selection operates in the long run to select investors whose beliefs about the economy are correct. Within this framework they provide conditions on risk preferences, time discount factors, and beliefs that result in survival. They make it clear that incorrect beliefs might persist in the long run, driving out investors who actually hold the correct beliefs. More recently, Sciubba (1999a, b) and Sandroni (2000) have provided additional conditions on market structure under which irrational investors will not survive.

For example, if a rational arbitrageur can convince rational investors that overreaction and underreaction are due to cognitive biases, then he will have access to capital allowing him to bet against such mispricing. If he is right, then he will earn superior returns for his investors. Competition for those returns will cause the anomaly to disappear, even if irrational investors are unable to learn their way out of their bad investment strategies. Of course, in a world with many arbitrageurs, it may matter how many others have reached the conclusion that anomalies are caused by cognitive biases and how large is the aggregate pool of capital that is available to them. Arbitrageurs may bet more or less aggressively against mispricing depending on their beliefs about the actions of other arbitrageurs. On the one hand, the more arbitrageurs with independent sources of capital identify mispricing, the faster prices should converge to the correct level, and the more willing should an arbitrageur be to bet against mispricing. On the other hand, arbitrageurs may bet less aggressively against mispricing either because they believe that many others have also identified the same anomaly and that expected returns to their position may be low or that the flow of capital to arbitrage activity is positively correlated across arbitrageurs. Also, arbitrageurs face the additional risk that irrational investors actually have learned their way out of the bad investment strategy that the arbitrageur sees in the historical data. In that case, current prices may no longer reflect the degree of mispricing needed to justify the strategy.

The limits of arbitrage will be especially severe if cognitive biases are themselves related to structural uncertainty. Earlier we suggested that the mathematical similarities of the theories raise the intriguing (though admittedly speculative) possibility that the cognitive biases we see are related to underlying structural uncertainty problems. That is, the same sorts of environments that present considerable challenges in dealing rationally with structural uncertainty may also create ideal conditions for the appearance of cognitive biases.

Consider the recent wave of Internet stock gains. Many investors believe that Internet stocks are overvalued and that this overvaluation is related to the naïve strategies of individual investors. Yet few of these arguably rational investors are willing to make large bets against the perceived mispricing, *even those with the capital to do so*. Anecdotal evidence, at least, suggests that structural uncertainty plays a role. Whatever the perceptions of investor irrationality, there is sufficient uncertainty about the underlying structural process (the future importance of the Internet, the likely survivors, etc.) that many are unwilling to place their bets. Were investor euphoria to attach to steel or oil stocks, it is difficult to believe that sophisticated investors would be as reluctant, since the structural uncertainty would be much lower. The very existence of Internet investor euphoria may be tied to the same sources of structural uncertainty than prevent its exploitation. The uncertainty surrounding so many Internet ventures may allow long periods of optimism

to survive, while at the same time preventing rational investors from learning to distinguish the future winners from the future losers.

In summary, the existence of competing theories links ideas of learning and arbitrage to the survival of financial anomalies. If rational structural uncertainty causes financial anomalies, then their disappearance hinges on the ability of the rational investors to become better calibrated to the structural features of the data. This is a nontrivial task in the short run, even if the economy's structural features remain stable. If those features are themselves changing, learning of this type may be impossible. Even if financial anomalies are caused by irrational investors, their disappearance still may hinge on the ability of rational investors to reject the competing rational explanation for observed price patterns. If rational arbitrageurs can be confident that mispricing exists (and can explain this to rational investors who provide their capital), then the arbitrage bounds created by standard limits of arbitrage arguments are likely to be quite small.<sup>27</sup>

#### **4. Conclusion**

In this article we have explored competing theories of financial anomalies—behavioral theories and rational theories—stressing the consequences of their opposite deviations from the rational expectations ideal. Our comparative analysis highlights the explanatory approaches of the two theories and the need to more carefully distinguish the theories from each other.

At the same time, our analysis also suggests that interesting connections may exist between behavioral and rational approaches that warrant greater study. The mathematical similarities between the behavioral and rational approaches are particularly intriguing. Consider the recent model of Barberis, Shleifer, and Vishny (1998). The authors interpret their model as capturing both the representativeness heuristic and conservatism, and there is no doubt that they intend for their representative investor to be interpreted in a behavioral sense. But (as they acknowledge) one fact about their model is striking: it is fully Bayesian so that the model's mathematical structure is, in a formal sense, consistent with rational information processing. Their results are driven by the fact that their representative investor holds the prior belief that the true model for earnings is impossible (not in the support of his prior over models). In an essentially isomorphic approach, however, Nyarko (1991) examined the monopolist's problem of learning a demand curve when the true parameters of the demand curve lie outside the support

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<sup>27</sup> Some mispricing may survive, however, if there are also legal or regulatory limits to arbitrage. For example, short sale constraints might explain the survival of overpricing [Morris (1996)], while prudent laws might explain the existence of underpricing for potentially "imprudent" investments like some value stocks [Del Guercio (1996)]. Such constraints have not been the focus of the limits of arbitrage literature, however, and little evidence currently exists to show that such constraints are strong enough to explain observed mispricing on their own.



of his prior distribution. He shows that—similar to the Barberis, Shleifer, and Vishny (1998) result—the monopolist would cycle indefinitely between two erroneous models that come closest to the true model, which by assumption he can never learn. However, Nyarko (1991) adopts a completely rational interpretation of his model. What is important in both approaches—in terms of delivering interesting testable predictions of economic behavior—is the structural uncertainty, not the philosophical characterization. Future work exploring the convergence of these approaches would be quite interesting, especially if it adds greater economic context to the assumptions adopted in behavioral finance.

One area not explored in detail here is the important normative differences that exist between the competing theories. Normative differences arise from the normative implications of the benchmark efficient markets, rational expectations asset pricing theory. Consider again the two main assumptions of that theory: complete rationality and complete structural knowledge of the economy. These assumptions are more than pillars of the traditional model, they are the best outcome possible. By definition, one can do no better than perfect rationality and full information. Thus prices in the efficient markets, rational expectations model are more than simple predicted quantities of a scientific model. Rather the price is “right” in such models, in a *normative* sense. Anomalous prices are “wrong” prices, implying the possibility of social gains if wrong prices are correctable. This raises obvious questions concerning whether financial anomalies justify government intervention in capital markets, and if so, what kind of intervention [see Shleifer (2000, chap. 7)]. While there may be a role for capital market regulatory policy in either case, the goals of legal intervention are likely to be quite different depending on which theory of financial anomalies dominates public debate—rational or behavioral.

For example, rational structural uncertainty models may imply a role for governmental intervention to improve information disclosure by firms. If the clues to structural knowledge lie within the firm, then mechanisms for forcing those clues into the market may be socially beneficial. Still, such regulations are likely to leave investment decision making relatively unfettered and unregulated, with restrictions on outright fraud, but few regulations restricting investment choice, per se. Investors in a rational world have little need for paternalistic capital market policy [see Stout (1997)]. Matters may be far different when investors are irrational. As Jolls, Sunstein, and Thaler (1998) note, behavioral approaches call into question the “strongly antipaternalistic” bias of traditional economic analysis of law. Certainly there are areas of U.S. securities law that are hard to understand unless concern with investor irrationality motivates at least some policy choices.

Of course, in choosing between legal regimes targeted at investor irrationality and legal regimes targeted at rational structural uncertainty, lawmakers and regulators are likely to face the same sorts of problems we highlighted in

our discussion of the limits of arbitrage. When it is easy to identify irrationality, as might occur with certain forms of fraud exploiting the most optimistic investors, legal rules are likely to be reasonably effective. But like an arbitrageur who cannot be sure enough of mispricing to make large bets against it, there may be times when it is impossible for even a rational regulator to determine whether events like stock market crashes reflected irrationality inviting government intervention or rational structural uncertainty best left alone.<sup>28</sup>

At a minimum, future work must focus on the interaction of rational and irrational investors. That work must start with the expectations formation of rational arbitrageurs (and their investors) in environments where irrationality might also exist. Neither paradigm can proceed indefinitely without a better understanding of this issue. Adherents of the rational approach (even the rational structural uncertainty approach) must be able to explain why the presence of irrational investors will not affect asset prices. Adherents of the behavioral approach need to explain the opposite. For a while at least, this common question provides a fruitful avenue for researchers from both approaches.

## Appendix A

Deriving the posterior probabilities of the change points requires that we first specify the joint posterior distribution and then integrate it with respect to  $\mu_A$ ,  $\mu_B$ , and  $\sigma^2$ . The likelihood function is proportional to

$$l(x_1, \dots, x_n | r, \mu_A, \mu_B, \sigma) \propto (\sigma)^{-n} \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot \left[\sum_{i=1}^r (x_i - \mu_A)^2 + \sum_{i=r+1}^n (x_i - \mu_B)^2\right]\right). \quad (A1)$$

Informative prior beliefs are specified as conditional normal distributions for  $\mu_A | \sigma^2$  and  $\mu_B | \sigma^2$  and a scaled inverse  $\chi^2$  for  $\sigma^2$ :

$$\begin{aligned} \mu_A | \sigma^2 &\sim N(\mu_0, \sigma^2 / \kappa_0) \\ \mu_B | \sigma^2 &\sim N(\mu_0, \sigma^2 / \kappa_0) \\ \sigma^2 &\sim \text{Inv} - \chi^2(v_0, \sigma_0^2). \end{aligned}$$

<sup>28</sup> Other normative areas are likely to manifest important differences as well. For example, one of the most important practical concerns of managers and financial analysts is the estimation of costs of capital. "True" costs of capital may be hard to infer from market prices in a behavioral theory [Stein (1996), Haugen (1999)], while adjustments to infer costs of capital in structural uncertainty models may be much easier [see, e.g., Mayfield (1999)].

Finally, we set a uniform prior for the possible change points  $r \in \{1, \dots, n\}$ . The joint posterior is therefore proportional to

$$(\sigma)^{-(n+\kappa_0+4)} \cdot \exp \left\{ -\frac{1}{2\sigma^2} \cdot \left[ \sum_{i=1}^r (x_i - \mu_A)^2 + \sum_{i=r+1}^n (x_i - \mu_B)^2 + \kappa_0((\mu_A - \mu_0)^2 + (\mu_B - \mu_0)^2) + \nu_0\sigma_0^2 \right] \right\}. \quad (A2)$$

The posterior probability of a change point  $r \in \{1, \dots, n\}$  is obtained by integrating Equation (A2) over the unknown parameters  $\mu_A$ ,  $\mu_B$ , and  $\sigma^2$ :

$$p_n(r=1, \dots, n-1) \propto \{(\kappa_0+r) \cdot (\kappa_0+n-r)\}^{-\frac{1}{2}} \cdot \left\{ \sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2 + \kappa_0 \left( \frac{r}{\kappa_0+r} (\bar{x}_r - \mu_0)^2 + \frac{(n-r)}{\kappa_0+n-r} (\bar{x}_{n-r} - \mu_0)^2 \right) + \nu_0\sigma_0^2 \right\}^{-\left(\frac{n+\kappa_0}{2}\right)} \quad (A3)$$

$$p_n(r=n) \propto (\kappa_0+n)^{-\frac{1}{2}} \cdot \left\{ \sum_{i=1}^n (x_i - \bar{x}_n)^2 + \kappa_0 \left( \frac{n}{\kappa_0+n} (\bar{x}_n - \mu_0)^2 \right) + \nu_0\sigma_0^2 \right\}^{-\left(\frac{n+\nu_0}{2}\right)}$$

where  $\bar{x}_r = r^{-1} \cdot \sum_{i=1}^r x_i$ ,  $\bar{x}_{n-r} = (n-r)^{-1} \cdot \sum_{i=r+1}^n x_i$ , and  $\bar{x}_n = n^{-1} \cdot \sum_{i=1}^n x_i$ .

It is instructive to examine these equations. First, the terms in the curly brackets. The first two terms,  $\sum_{i=1}^r (x_i - \bar{x}_r)^2$  and  $\sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2$ , should be low if a change has occurred at  $r$ . This leads the whole term to be large relative to other possible changes. The next two terms,  $\frac{r}{\kappa_0+r} (\bar{x}_r - \mu_0)^2$  and  $\frac{(n-r)}{\kappa_0+n-r} (\bar{x}_{n-r} - \mu_0)^2$ , penalize the in-sample fit if it deviates from the investor's prior beliefs  $\mu_0$  and lead to a lower posterior probability of a change. The last term,  $\nu_0\sigma_0^2$ , reflects investor's prior uncertainty and is shared across all change-point scenarios. Therefore a higher prior variance  $\sigma_0^2$  implies that, a priori, it is going to be harder to distinguish across the different posterior probabilities of a change. Finally, the terms which premultiply the curly brackets,

$$[(\kappa_0+r) \cdot (\kappa_0+n-r)]^{-\frac{1}{2}} \quad \forall r = 1, \dots, n-1,$$

deserve attention as well. Their effect is to increase the posterior probability of a change occurring at a low or high index of change  $r$  relative to a change occurring midway through the sample. This results from the fact that, given the independence of the sample realizations, a longer period over which any one subsample is measured yields more precise inferences regarding the in-sample mean, which more than outweighs the adjacent and shorter subperiod. Consequently a break at  $r=2$  is favored over a break occurring at  $r=10$  simply because the overall precision based on  $\bar{x}_2$  and  $\bar{x}_{n-2}$  is higher than the precision for  $\bar{x}_{10}$  and  $\bar{x}_{n-10}$ .

## Appendix B

The mean of the posterior distribution for  $\mu_n$  is given as a mixture of the individual posterior means conditioned on each of the possible change points. Each of these in turn is a simple posterior mean of a Student's  $t$ -distribution [see Section 1.4 and Equation (2)].

$$\begin{aligned}
 \hat{\mu}_n &= E \left[ \sum_{r=1}^{n-1} p_n(\mu_B|r) p_n(r) + p_n(\mu_A|r=n) p_n(n) \right] \\
 &= p_n(1) \left[ \frac{(n-1)}{(n-1) + \kappa_0} \bar{x}_{n-1} + \frac{\kappa_0}{(n-1) + \kappa_0} \mu_0 \right] + p_n(2) \left[ \frac{(n-2)}{(n-2) + \kappa_0} \bar{x}_{n-2} + \frac{\kappa_0}{(n-2) + \kappa_0} \mu_0 \right] \\
 &\quad + \dots + p_n(n-1) \left[ \frac{n-(n-1)}{(n-(n-1)) + \kappa_0} \bar{x}_{n-(n-1)} + \frac{\kappa_0}{(n-(n-1)) + \kappa_0} \mu_0 \right] \\
 &\quad + p_n(n) \left[ \frac{n}{n + \kappa_0} \bar{x}_n + \frac{\nu_0}{n + \kappa_0} \mu_0 \right] \\
 &= \sum_{i=1}^{n-1} p_n(i) \left[ \frac{(n-i)}{(n-i) + \kappa_0} \bar{x}_{n-i} + \frac{\nu_0}{(n-i) + \kappa_0} \mu_0 \right] + p_n(n) \left[ \frac{n}{n + \kappa_0} \bar{x}_n + \frac{\kappa_0}{n + \kappa_0} \mu_0 \right],
 \end{aligned}$$

where  $\bar{x}_{n-i}$  denotes the mean of the last  $n-i$  observations (all observations after the change point on which the mean is conditioned) and the  $p_n(i)$  are as defined in Equation (7).

### Appendix C

This appendix proves that the estimator in Equation (8) gives more weight to recent data than old data. Consider Equation (8):

$$\hat{\mu}_n = \sum_{i=1}^{n-1} p_n(i) \left[ \frac{(n-i)}{(n-i) + \nu_0} \bar{x}_{n-i} + \frac{\nu_0}{(n-i) + \nu_0} \mu_0 \right] + p_n(n) \left[ \frac{n}{n + \nu_0} \bar{x}_n + \frac{\nu_0}{n + \nu_0} \mu_0 \right]. \quad (A4)$$

The data enter the estimator through the sample means,  $\bar{x}_{n-i}$  and  $\bar{x}_n$ . The proof involves showing that recent data enters more of these sample means than old data, and receives more weight in the estimator. Consider first the third term. This term captures the possibility in the investor's mind that there was no change from  $\mu_A$  to  $\mu_B$ . Therefore all  $n$  data points enter the sample mean,  $\bar{x}_n$ , in that term. Each of the  $n$  data points receives weight,

$$p_n(n) \left( \frac{1}{n} \right) \left( \frac{n}{n + \nu_0} \right), \quad (A5)$$

from the sample mean that enters that term. This weight is made up of three parts. First, the posterior probability of the "no change" point  $r = n$ . Second, the term  $1/n$  in the calculation of the sample mean for  $n$  observations. Third, the weight on the sample mean when there are  $n$  observations and the informative prior.

A data point enters the other sample means  $\bar{x}_{n-i}$  only if it occurs after the change point  $i$ . For example, if  $i = 2$ , the sample mean  $\bar{x}_{n-2}$  includes only observations occurring after the second observation (since, by definition, this means that  $\mu_A$  generated the first two observations, then a change occurred, and  $\mu_B$  generated all remaining observations). Thus each data point  $j \in \{2, \dots, n\}$  (remembering that the only weight the first observation can receive is the weight if no change occurred) also receives the sum of the weights associated with each sample mean that includes that observation. For a given data point  $j$ , this weight is given by

$$\sum_{k=1}^{j-1} p_n(k) \left( \frac{1}{n-k} \right) \left( \frac{n-k}{(n-k) + \nu_0} \right). \quad (A6)$$

Consider, for example, observation 4. There are three terms in the sum of the weight on observation 4 conditional on some change ( $k = 1$  to 3). First, the change could have occurred at observation 1, in which case observation 4 will enter into the sample mean constructed from all observations after the first. Second, the change could have occurred at observation 2, in which

case observation 4 will enter into the sample mean constructed from all observations after the second. Finally, the change could have occurred at observation 3, in which case observation 4 will enter into the sample mean constructed from all observations after the third. After that, however, the change will take place at or after observation 4, so observation 4 cannot enter the calculation of the sample mean.

Combining the weights that arise from change and no change, we get the weight placed on observation  $j \in \{2, \dots, n\}$  in the estimator:

$$\sum_{k=1}^{j-1} p_n(k) \left( \frac{1}{n-k} \right) \left( \frac{n-k}{(n-k)+\nu_0} \right) + p_n(n) \left( \frac{1}{n} \right) \left( \frac{n}{n+\nu_0} \right). \tag{A7}$$

This sum is increasing in  $j$ , while the weight on observation  $j = 1$  is simply  $p_n(n) \left( \frac{1}{n} \right) \left( \frac{n}{n+\nu_0} \right)$ . This completes the proof.

## Appendix D

This appendix proves that the rational structural uncertainty model can generate an “excessive certainty” or “overconfidence” effect in environments characterized by stability of the valuation relevant parameter.

Assume that the priors on the model parameters are diffuse:<sup>29</sup>

$$p(\mu_A, \mu_B, \sigma^2) \propto \sigma^{-2} d\mu_A d\mu_B d\sigma.$$

The prior beliefs on the possible change points  $r \in \{1, \dots, n\}$  are diffuse as in Section 1. The posterior probability of a change at point  $r$ , denoted  $p_n(r)$ , is obtained by integrating the likelihood function with respect to the unknown mean and precision parameters and is proportional to

$$p_n(r = 1, \dots, n-1) \propto \{r \cdot (n-r)\}^{-\frac{1}{2}} \cdot \left\{ \sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2 \right\}^{-\left(\frac{n}{2}-1\right)} \tag{A8}$$

$$p_n(r = n) \propto n^{-\frac{1}{2}} \cdot \left\{ \sum_{i=1}^n (x_i - \bar{x}_n)^2 \right\}^{-\frac{n-1}{2}},$$

where  $\bar{x}_r = r^{-1} \cdot \sum_{i=1}^r x_i$ ,  $\bar{x}_{n-r} = (n-r)^{-1} \cdot \sum_{i=r+1}^n x_i$ , and  $\bar{x}_n = n^{-1} \cdot \sum_{i=1}^n x_i$ .

Next we show that the estimates of the sample variance for the investor concerned with instability will be lower relative to the investor who believes in stability (regardless of who is correct). To do so, we now derive the posterior distribution of the variance,  $p_n(\sigma^2)$ , for both of these investors. First, note that this posterior distribution, conditioned on a break occurring at  $r (1 \leq r \leq n)$ , is in the form of a scaled inverse  $\chi^2$ :

$$p_n(\sigma^2 | r=1, \dots, n-1) \propto \sigma^{-n} \cdot \exp \left\{ -\frac{(n-2)}{2\sigma^2} \cdot \left[ \sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2 \right] / (n-2) \right\} \tag{A9}$$

$$p_n(\sigma^2 | r=n) \propto \sigma^{-(n+1)} \cdot \exp \left\{ -\frac{(n-1)}{2\sigma^2} \cdot \sum_{i=1}^n (x_i - \bar{x}_n)^2 / (n-1) \right\}.$$

<sup>29</sup> We use uninformative priors to show how beliefs regarding instability interact with the sample information alone, leading to systematically lower posterior beliefs for  $\sigma^2$  compared to a belief in stability.

The posterior distribution  $p_n(\sigma^2)$ , which integrates over the uncertainty regarding the possible change point, is just the weighted average of these conditional distributions using the posterior probabilities of a change  $p_n(r)$  as weights. Consequently the posterior mean of  $p_n(\sigma^2)$ , for the investor concerned with instability,  $\hat{\sigma}_{unstable}^2$ , is given by the following expression:

$$\hat{\sigma}_{unstable}^2 = \sum_{r=1}^{n-1} p_n(r) \cdot \left( \frac{[\sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2]}{n-4} \right) + p_n(r=n) \cdot \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{n-3} \tag{A10}$$

Note, however, that the investor with a belief in stability has posterior beliefs regarding the variance in the form of a scaled inverse  $\chi^2$  as well:

$$p_n(\sigma^2) \propto \sigma^{-(n+1)} \cdot \exp \left\{ -\frac{(n-1)}{2\sigma^2} \cdot \sum_{i=1}^n (x_i - \bar{x}_n)^2 / (n-1) \right\}, \tag{A11}$$

and the posterior mean, denoted by  $\hat{\sigma}_{stable}^2$ , is given by

$$\hat{\sigma}_{stable}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x}_n)^2}{n-3}. \tag{A12}$$

Comparison of the two posterior means  $\hat{\sigma}_{unstable}^2$  and  $\hat{\sigma}_{stable}^2$  [Equations (A10) and (A12)] is directly related to the following inequality:

$$\sum_{i=1}^r (x_i - \bar{x}_r)^2 + \sum_{i=r+1}^n (x_i - \bar{x}_{n-r})^2 \leq \sum_{i=1}^n (x_i - \bar{x}_n)^2,$$

which holds strictly whenever the subsample means  $(\bar{x}_r, \bar{x}_{n-r})$  are not equal to the overall sample mean  $\bar{x}_n$ ,  $\forall r = 1, \dots, n-1$ . The investor who believes in stability calculates his measure of variability relative to the overall sample mean. However, an investor concerned with a possible change point will derive his measure of variability relative to what he considers to be the changing sample means, which necessarily leads to a lower overall sum of squares.

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